Nonstationary Time Series Data and Cointegration

Chapter 12

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Chapter 12: Nonstationary Time Series Data and Cointegration

- 12.1 Stationary and Nonstationary Variables
- 12.2 Spurious Regressions
- 12.3 Unit Root Tests for Stationarity
- 12.4 Cointegration
- 12.5 Regression When There is No Cointegration

12.1 Stationary and Nonstationary Variables

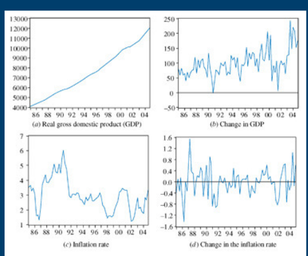

Figure 12.1(a) US economic time series

12.1 Stationary and Nonstationary Variables

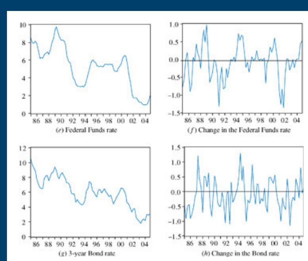

Figure 12.1(b) US economic time series

12.1 Stationary and Nonstationary Variables

$$E(y_t) = \mu \quad (12.1a)$$

$$\text{var}(y_t) = \sigma^2 \quad (12.1b)$$

$$\text{cov}(y_t, y_{t+s}) = \text{cov}(y_t, y_{t-s}) = \gamma_s \quad (12.1c)$$

12.1 Stationary and Nonstationary Variables

Table 12.1 Sample Means of Time Series Shown in Figure 12.1

Variable	Sample periods 1985:1 to 1994:4	Sample periods 1995:1 to 2004:4
Real GDP (a)	5587.7	9465.4
Inflation rate (c)	3.5	2.4
Federal Funds rate (e)	6.3	4.1
Bond rate (g)	7.2	4.7
Change in GDP (b)	79.9	119.1
Change in the inflation rate (d)	−0.03	0.02
Change in the Federal Fund rate (f)	−0.1	−0.1
Change in the Bond rate (h)	−0.1	−0.1

12.1.1 The First-Order Autoregressive Model

$$y_t = \rho y_{t-1} + v_t, \qquad |\rho| < 1 \quad (12.2a)$$

$$y_1 = \rho y_0 + v_1$$

$$y_2 = \rho y_1 + v_2 = \rho(\rho y_0 + v_1) + v_2 = \rho^2 y_0 + \rho v_1 + v_2$$

$$\vdots$$

$$y_t = v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \ldots.. + \rho^t y_0$$

12.1.1 The First-Order Autoregressive Model

$$E[y_t] = E[v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \ldots..] = 0$$

$$(y_t - \mu) = \rho(y_{t-1} - \mu) + v_t$$

$$y_t = \alpha + \rho y_{t-1} + v_t, \qquad |\rho| < 1 \quad (12.2b)$$

12.1.1 The First-Order Autoregressive Model

$$E(y_t) = \mu = \alpha/(1-\rho) = 1/(1-0.7) = 3.33$$

$$(y_t - \mu - \delta t) = \rho(y_{t-1} - \mu - \delta(t-1)) + v_t, \quad |\rho| < 1$$

$$y_t = \alpha + \rho y_{t-1} + \lambda t + v_t \quad (12.2c)$$

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12.1.1 The First-Order Autoregressive Model

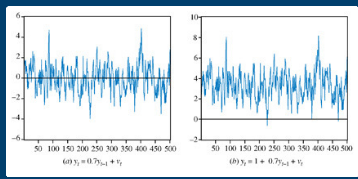

Figure 12.2 (a) Time Series Models

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12.1.1 The First-Order Autoregressive Model

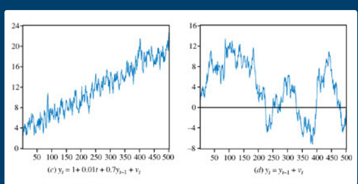

Figure 12.2 (b) Time Series Models

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12.1.1 The First-Order Autoregressive Model

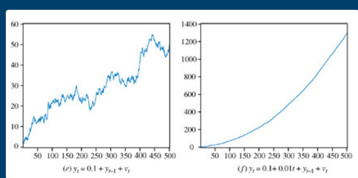

Figure 12.2 (c) Time Series Models

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12.1.2 Random Walk Models

$$y_t = y_{t-1} + v_t \qquad (12.3a)$$

$$y_1 = y_0 + v_1$$

$$y_2 = y_1 + v_2 = (y_0 + v_1) + v_2 = y_0 + \sum_{s=1}^{2} v_s$$

$$\vdots$$

$$y_t = y_{t-1} + v_t = y_0 + \sum_{s=1}^{t} v_s$$

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12.1.2 Random Walk Models

$$E(y_t) = y_0 + E(v_1 + v_2 + \ldots + v_t) = y_0$$

$$\text{var}(y_t) = \text{var}(v_1 + v_2 + \ldots + v_t) = t\sigma_v^2$$

$$y_t = \alpha + y_{t-1} + v_t \qquad (12.3b)$$

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12.1.2 Random Walk Models

$$y_1 = \alpha + y_0 + v_1$$

$$y_2 = \alpha + y_1 + v_2 = \alpha + (\alpha + y_0 + v_1) + v_2 = 2\alpha + y_0 + \sum_{s=1}^{2} v_s$$

$$\vdots$$

$$y_t = \alpha + y_{t-1} + v_t = t\alpha + y_0 + \sum_{s=1}^{t} v_s$$

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12.1.2 Random Walk Models

$$E(y_t) = t\alpha + y_0 + E(v_1 + v_2 + v_3 + \ldots + v_t) = t\alpha + y_0$$

$$\text{var}(y_t) = \text{var}(v_1 + v_2 + v_3 + \ldots + v_t) = t\sigma_v^2$$

$$y_t = \alpha + \delta t + y_{t-1} + v_t \qquad (12.3c)$$

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12.1.2 Random Walk Models

$$y_1 = \alpha + \delta + y_0 + v_1; \quad (\text{since } t = 1)$$

$$y_2 = \alpha + \delta 2 + y_1 + v_2 = \alpha + 2\delta + (\alpha + \delta + y_0 + v_1) + v_2 = 2\alpha + 3\delta + y_0 + \sum_{s=1}^{2} v_s$$

$$\vdots$$

$$y_t = \alpha + \delta t + y_{t-1} + v_t = t\alpha + \left(\frac{t(t+1)}{2}\right)\delta + y_0 + \sum_{s=1}^{t} v_s$$

$$1 + 2 + 3 + \cdots t = t(t+1)/2$$

12.2 Spurious Regressions

$$rw_1: \; y_t = y_{t-1} + v_{1t}$$

$$rw_2: \; x_t = x_{t-1} + v_{2t}$$

$$\hat{rw}_{1t} = 17.818 + 0.842 \, rw_{2t}, \quad R^2 = .70$$
$$(t) \qquad (40.837)$$

12.2 Spurious Regressions

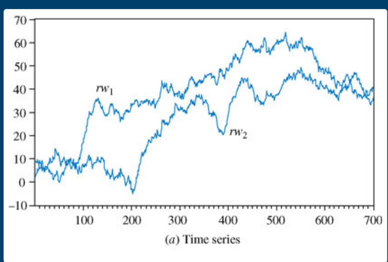

Figure 12.3 (a) Time Series of Two Random Walk Variables

12.2 Spurious Regressions

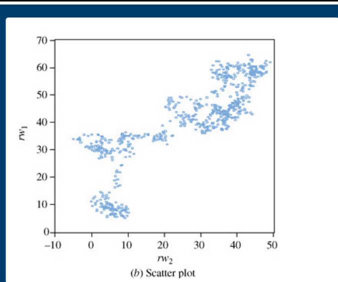

Figure 12.3 (b) Scatter Plot of Two Random Walk Variables

12.3 Unit Root Test for Stationarity

- 12.3.1 Dickey-Fuller Test 1 (no constant and no trend)

$$y_t = \rho y_{t-1} + v_t \quad (12.4)$$

$$y_t - y_{t-1} = \rho y_{t-1} - y_{t-1} + v_t$$

$$\Delta y_t = (\rho - 1) y_{t-1} + v_t \quad (12.5a)$$

$$= \gamma y_{t-1} + v_t$$

12.3 Unit Root Test for Stationarity

- 12.3.1 Dickey-Fuller Test 1 (no constant and no trend)

$$H_0 : \rho = 1 \quad \leftrightarrow \quad H_0 : \gamma = 0$$

$$H_1 : \rho < 1 \quad \leftrightarrow \quad H_1 : \gamma < 0$$

12.3 Unit Root Test for Stationarity

- 12.3.2 Dickey-Fuller Test 2 (with constant but no trend)

$$\Delta y_t = \alpha + \gamma y_{t-1} + v_t \quad (12.5b)$$

12.3 Unit Root Test for Stationarity

- 12.3.3 Dickey-Fuller Test 3 (with constant and with trend)

$$\Delta y_t = \alpha + \gamma y_{t-1} + \lambda t + v_t \quad (12.5c)$$

12.3.4 The Dickey-Fuller Testing Procedure

First step: plot the time series of the original observations on the variable.

- If the series appears to be wandering or fluctuating around a sample average of zero, use test equation (12.5a).
- If the series appears to be wandering or fluctuating around a sample average which is non-zero, use test equation (12.5b).
- If the series appears to be wandering or fluctuating around a linear trend, use test equation (12.5c).

12.3.4 The Dickey-Fuller Testing Procedure

Table 12.2 Critical Values for the Dickey–Fuller Test

Model	1%	5%	10%
$\Delta y_t = \gamma y_{t-1} + v_t$	−2.56	−1.94	−1.62
$\Delta y_t = \alpha + \gamma y_{t-1} + v_t$	−3.43	−2.86	−2.57
$\Delta y_t = \alpha + \lambda t + \gamma y_{t-1} + v_t$	−3.96	−3.41	−3.13
Standard critical values	−2.33	−1.65	−1.28

12.3.4 The Dickey-Fuller Testing Procedure

- An important extension of the Dickey-Fuller test allows for the possibility that the error term is autocorrelated.

$$\Delta y_t = \alpha + \gamma y_{t-1} + \sum_{s=1}^{m} a_s \Delta y_{t-s} + v_t \quad (12.6)$$

$$\Delta y_{t-1} = (y_{t-1} - y_{t-2}),\ \Delta y_{t-2} = (y_{t-2} - y_{t-3}), \ldots$$

- The unit root tests based on (12.6) and its variants (intercept excluded or trend included) are referred to as **augmented Dickey-Fuller tests**.

12.3.5 The Dickey-Fuller Tests: An Example

$$\widehat{\Delta F_t} = 0.178 - 0.037 F_{t-1} + 0.672 \Delta F_{t-1}$$
$$(tau) \qquad (-2.090)$$

$$\widehat{\Delta B_t} = 0.285 - 0.056 B_{t-1} + 0.315 \Delta B_{t-1}$$
$$(tau) \qquad (-1.976)$$

12.1 Stationary and Nonstationary Variables

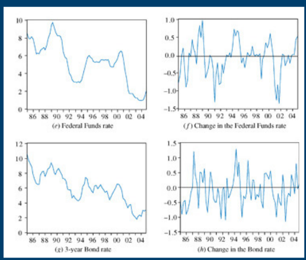

Figure 12.1(b) US economic time series

12.3.6 Order of Integration

$$\Delta y_t = y_t - y_{t-1} = v_t$$

$$\widehat{\Delta(\Delta F)}_t = -0.340(\Delta F)_{t-1}$$

$$(tau) \qquad (-4.007)$$

$$\widehat{\Delta(\Delta B)}_t = -0.679(\Delta B)_{t-1}$$

$$(tau) \qquad (-6.415)$$

12.4 Cointegration

$$\Delta\hat{e}_t = \gamma\hat{e}_{t-1} + v_t \qquad (12.7)$$

$$Case\,1:\ \ \hat{e}_t = y_t - bx_t \qquad (12.8a)$$

$$Case\,2:\ \ \hat{e}_t = y_t - b_2x_t - b_1 \qquad (12.8b)$$

$$Case\,3:\ \ \hat{e}_t = y_t - b_2x_t - b_1 - \hat{\delta}t \qquad (12.8c)$$

12.4 Cointegration

Table 12.3 **Critical Values for the Cointegration Test**

Regression model	1%	5%	10%
(1) $y_t = \beta x_t + e_t$	−3.39	−2.76	−2.45
(2) $y_t = \beta_1 + \beta_2 x_t + e_t$	−3.96	−3.37	−3.07
(3) $y_t = \beta_1 + \delta t + \beta_2 x_t + e_t$	−3.98	−3.42	−3.13

12.4.1 An Example of a Cointegration Test

$$\hat{B}_t = 1.644 + 0.832F_t, \quad R^2 = 0.881$$
$$(t) \quad (8.437) \quad (24.147) \qquad (12.9)$$

$$\Delta\hat{e}_t = -0.314\hat{e}_{t-1} + 0.315\Delta\hat{e}_{t-1}$$
$$(tau) \quad (-4.543)$$

12.4.1 An Example of a Cointegration Test

The null and alternative hypotheses in the test for cointegration are:

H_0 : the series are not cointegrated ⇔ residuals are nonstationary

H_1 : the series are cointegrated ⇔ residuals are stationary

12.5 Regression When There Is No Cointegration

- **12.5.1 First Difference Stationary**

$$y_t = y_{t-1} + v_t$$

$$\Delta y_t = y_t - y_{t-1} = v_t$$

The variable y_t is said to be a **first difference stationary** series.

12.5.1 First Difference Stationary

$$\Delta y_t = \theta\Delta y_{t-1} + \beta_0\Delta x_t + \beta_1\Delta x_{t-1} + e_t \qquad (12.10a)$$

$$y_t = \alpha + y_{t-1} + v_t$$

$$\Delta y_t = \alpha + v_t$$

$$\Delta y_t = \alpha + \theta\Delta y_{t-1} + \beta_0\Delta x_t + \beta_1\Delta x_{t-1} + e_t \qquad (12.10b)$$

12.5.2 Trend Stationary

$$y_t = \alpha + \delta t + v_t$$

$$y_t - \alpha - \delta t = v_t$$

$$y_t^* = \theta y_{t-1}^* + \beta_0 x_t^* + \beta_1 x_{t-1}^* + e_t \quad (12.11)$$

$$y_t = \alpha + \delta t + \theta y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + e_t$$

where $\alpha = \alpha_1(1-\theta_1) - \alpha_2(\beta_0 + \beta_1) + \theta_1\delta_1 + \beta_1\delta_2$

and $\delta = \delta_1(1-\theta_1) - \delta_2(\beta_0 + \beta_1)$

12.5.2 Trend Stationary

To summarize:

- If variables are stationary, or I(1) and cointegrated, we can estimate a regression relationship between the levels of those variables without fear of encountering a spurious regression.
- If the variables are I(1) and not cointegrated, we need to estimate a relationship in first differences, with or without the constant term.
- If they are trend stationary, we can either de-trend the series first and then perform regression analysis with the stationary (de-trended) variables or, alternatively, estimate a regression relationship that includes a trend variable. The latter alternative is typically applied.

Keywords

- Augmented Dickey-Fuller test
- Autoregressive process
- Cointegration
- Dickey-Fuller tests
- Mean reversion
- Order of integration
- Random walk process
- Random walk with drift
- Spurious regressions
- Stationary and nonstationary
- Stochastic process
- Stochastic trend
- Tau statistic
- Trend and difference stationary
- Unit root tests